

# The largest values of Dedekind sums

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## Abstract

Let  $s(m, n)$  denote the classical Dedekind sum, where  $n$  is a positive integer and  $m \in \{0, 1, \dots, n-1\}$ ,  $(m, n) = 1$ . For a given positive integer  $k$ , we describe a set of at most  $k^2$  numbers  $m$  for which  $s(m, n)$  may be  $\geq s(k, n)$ , provided that  $n$  is sufficiently large. For the numbers  $m$  not in this set,  $s(m, n) < s(k, n)$ .

## 1. Introduction and results

Let  $m$  and  $n$  be integers,  $n \neq 0$  and  $(m, n) = 1$ . The classical Dedekind sum  $s(m, n)$  is defined by

$$s(m, n) = \sum_{k=1}^{|n|} ((k/n))((mk/n))$$

where  $((\dots))$  is the “sawtooth function” defined by

$$((t)) = \begin{cases} t - [t] - 1/2 & \text{if } t \in \mathbb{R} \setminus \mathbb{Z}; \\ 0 & \text{if } t \in \mathbb{Z} \end{cases}$$

(see, for instance, [12, p. 1]).

In the present setting it is more convenient to work with

$$S(m, n) = 12s(m, n)$$

instead. Since  $S(m, -n) = S(m, n)$  and  $S(m+n, n) = S(m, n)$ , we obtain all Dedekind sums if we restrict  $n$  to positive integers and  $m$  to the range  $0 \leq m < n$ . The general case, however, will be needed below (see (6)).

The original context of Dedekind sums is the theory of modular forms (see [2]). But these sums have also interesting applications in connection with class numbers, lattice point problems, topology, and algebraic geometry (see [3, 10, 12, 13]). Starting with Rademacher [11], several authors have studied the distribution of Dedekind sums (for instance, [4, 6, 8, 14]). Whereas the arithmetic mean of the absolute values  $|S(m, n)|$ ,  $0 \leq m < n$ ,  $(m, n) = 1$ , has order of magnitude  $\log^2 n$  for  $n$  tending to infinity (see [7]), large Dedekind sums  $S(m, n)$  have order of magnitude  $n$ .

In this paper we study the largest values of Dedekind sums  $S(m, n)$  for a given sufficiently large number  $n$ .

In 1956, Rademacher showed

$$S(1, n) > S(m, n) \text{ for all } m, 2 \leq m \leq n-1, (m, n) = 1 \quad (1)$$

(see [11, Satz 2]). By the reciprocity law for Dedekind sums (see [12, p. 5]),

$$S(k, n) = -S(n, k) + \frac{n}{k} + \frac{k}{n} + \frac{1}{kn} - 3, \quad (2)$$

we obtain

$$S(1, n) = \frac{n^2 - 3n + 2}{n}. \quad (3)$$

So this largest of all Dedekind sums  $S(m, n)$  equals  $n + O(1)$  for  $n$  tending to infinity. Other large Dedekind sums are  $S(k, n)$  for a fixed integer  $k > 1$  and large numbers  $n$ . In fact,  $S(k, n) = n/k + O(1)$ , see (8). The main result of this paper is the following theorem.

**Theorem 1** *Let  $k$  be a positive integer. For sufficiently large integers  $n > k$  with  $(k, n) = 1$  and  $m \in \{0, \dots, n-1\}$ ,  $(m, n) = 1$ , we have*

$$S(m, n) \geq S(k, n)$$

*only if  $m$  has the form*

$$m = \frac{nc + q}{d}, \quad (4)$$

*with*

$$d \in \{1, \dots, k\}, c \in \{0, 1, \dots, d-1\}, (c, d) = 1, q \in \{1, \dots, \lfloor k/d \rfloor\}. \quad (5)$$

*Remarks.* 1. The proof of Theorem 1 shows

$$S(m, n) = \frac{n}{dq} + O(1)$$

for each of the numbers (4) in question. Since  $dq \leq k$ , we see that  $S(m, n) > S(k, n)$  whenever  $dq < k$ , whereas  $S(m, k) \geq S(m, n)$  may hold if  $dq = k$ . The proof of Theorem 1 also gives

$$S(m, n) \leq \frac{n}{k+1} + O(1)$$

for all numbers  $m$  not of the form described by (4) and (5).

2. It is easy to see that there are at most

$$\sum_{d=1}^k \varphi(d) \frac{k}{d}.$$

numbers  $m$  as described by (4), (5). This bound is  $\leq k^2$  or, more precisely,  $= 6k^2/\pi^2 + O(k \log k)$  (see [1, p. 70, Ex. 5]). In most cases, however, this is only a rough upper bound.

*Examples.* 1. For  $k = 3$ , the numbers  $m$  described by (4), (5) are  $m = 1, 2, 3$  (with  $d = 1, c = 0, q = 1, 2, 3$ ),  $m = (n+1)/2$  (with  $d = 2, c = 1, q = 1$ ), and  $m = (n+1)/3, (2n+1)/3$  (with  $d = 3, c = 1, 2, q = 1$ ). If  $(n+1)/2$  is an integer,  $n$  must be odd. Then  $S((n+1)/2, n) = S(2, n)$ , since  $(n+1)/2$  is the inverse of 2 mod  $n$ . The last two cases occur only if  $n \equiv 2 \pmod{3}$  and  $n \equiv 1 \pmod{3}$ , respectively. In each of these cases,  $S(m, n) = S(3, n)$ .

2. This example might suggest that for the numbers  $m$  given by (4), (5) the Dedekind sums  $S(m, n)$  take one of the values  $S(j, n)$ ,  $j \in \{1, \dots, k\}$ . This, however, is not true. Indeed, let  $k = 6$ ,  $d = 3$ ,  $c = 1$ ,  $q = 2$ , and  $n \geq 7$ , so  $m = (n + 2)/3$ . Since  $m$  must be an integer, we require  $n \equiv 1 \pmod{3}$ . Because  $k = 6$ ,  $n$  must be odd, and so  $n \equiv 1 \pmod{6}$ . Under this condition, we obtain from (6) below

$$S(m, n) = \frac{n^2 - 14n + 13}{6n},$$

whereas the reciprocity law (2) yields

$$S(6, n) = \frac{n^2 - 38n + 37}{6n}.$$

Accordingly,  $S(m, n) = S(6, n) + O(1)$ , but always  $S(m, n) > S(6, n)$ .

All terms  $O(1)$  in this paper can be transformed into explicit bounds. In this way, one may obtain results of Rademacher type (see (1)) for any given  $k$ . As an example, we settle the case  $k = 2$  here.

**Theorem 2** *Let  $n \geq 3$  be odd (hence  $S(2, n)$  is defined). Then for all  $m \in \{3, \dots, n-1\}$ ,  $(m, n) = 1$ , different from  $(n+1)/2$ ,*

$$S(2, n) > S(m, n).$$

## 2. Proofs

*Proof of Theorem 1.* Put  $l = 2k + 2$  and let  $n > l$ . We call a number  $m \in \{1, \dots, n-1\}$ ,  $(m, n) = 1$ , *ordinary* if, and only if, for all  $d \in \{1, \dots, l\}$  and all  $c \in \{0, 1, \dots, d\}$  with  $(c, d) = 1$ ,

$$\left| \frac{m}{n} - \frac{c}{d} \right| \geq \frac{l}{nd},$$

i.e., each possible  $q = md - nc$  satisfies  $|q| \geq l$ . Let  $m$  be an ordinary number. By a theorem about Farey approximation (see [9, p. 127, Th. 10.5]), there is a number  $d \in \{1, \dots, l\}$  and a  $c \in \{0, 1, \dots, d\}$ ,  $(c, d) = 1$ , such that

$$\left| \frac{m}{n} - \frac{c}{d} \right| \leq \frac{1}{ld}.$$

If we choose  $d$  and  $c$  in this way, we have  $|q| \leq n/l$  for the above  $q$ . Altogether,

$$l \leq |q| \leq \frac{n}{l}.$$

By [5, Lemma 1],

$$S(m, n) = S(c, d) + \varepsilon S(r, q) + \frac{n}{dq} + \frac{d}{nq} + \frac{q}{nd} - 3\varepsilon \quad (6)$$

where  $r$  is some integer prime to  $q$  and  $\varepsilon \in \{\pm 1\}$  is the sign of  $q$  (observe  $q \neq 0$  since  $n > d$ ). Combined with (3), this gives

$$|S(m, n)| \leq d + |q| + \frac{n}{d|q|} + \frac{d}{n|q|} + \frac{|q|}{nd} + 3.$$

We observe  $d \leq l$  and  $|q| \leq n/l$ . Further, since  $|q| \geq l$ , we have  $n/(d|q|) \leq n/l$ . The condition  $|q| \geq l \geq d$  implies  $d/(n|q|) \leq 1/n$ . From  $|q| \leq n/l$  we obtain  $|q|/(nd) \leq 1/l$ . Altogether,

$$|S(m, n)| \leq l + \frac{n}{l} + \frac{n}{l} + \frac{1}{n} + \frac{1}{l} + 3 = \frac{2n}{l} + l + \frac{1}{n} + \frac{1}{l} + 3 = \frac{n}{k+1} + O(1). \quad (7)$$

Next we show

$$S(k, n) = \frac{n}{k} + O(1). \quad (8)$$

To this end we observe  $S(n, k) \leq S(1, k)$  and  $S(n, k) \geq -S(1, k)$ , by (1). Then the reciprocity law (2), combined with (3), gives

$$S(k, n) \geq \frac{n^2 - (k^2 + 2)n + k^2 + 1}{kn} \text{ and } S(k, n) \leq \frac{n^2 + (k^2 - 6k + 2)n + k^2 + 1}{kn}.$$

This implies (8). Moreover, (7) and (8) show

$$S(m, n) < S(k, n)$$

for large numbers  $n$  and ordinary numbers  $m$ .

Now suppose that  $m$  is not an ordinary number. Therefore, there is a  $d \in \{1, \dots, l\}$  and a  $c \in \{0, 1, \dots, d\}$ ,  $(c, d) = 1$ , such that  $q = md - nc$  satisfies  $|q| < l$ . From (6) we obtain

$$S(m, n) = \frac{n}{dq} + O(1), \quad (9)$$

where

$$|O(1)| \leq d + |q| + \frac{d}{n|q|} + \frac{|q|}{nd} + 3 \leq 2l + \frac{2l}{n} + 3 \leq 2l + 5 \quad (10)$$

because  $n \geq l$ . Accordingly, if  $d|q| \geq k + 1$ , then  $|S(m, n)| < S(k, n)$  for large numbers  $n$ . Thus, the only numbers  $m$  to be considered are those with  $d|q| \leq k$ . They are, however, only of interest if  $q > 0$ , since, otherwise,  $S(m, n) < 0$  according to (9). But these numbers are just those described by (4), (5).  $\square$

*Proof of Theorem 2.* According to (7), we have, for  $k = 2$  and ordinary numbers  $m$ ,

$$|S(m, n)| \leq \frac{n}{3} + O(1),$$

with  $|O(1)| \leq 6 + 1/3 + 1/6 + 3 = 19/2$  since  $n \geq 3$ . Therefore, if  $n/3 + 19/2 < S(2, n) = (n^2 - 6n + 5)/2n$ , then  $|S(m, n)| < S(2, n)$ . This is the case for  $n \geq 75$ .

On the other hand, if  $m$  is not an ordinary number, (9) and (10) give

$$S(m, n) = \frac{n}{dq} + O(1),$$

with  $|O(1)| \leq 2l + 2l/n + 3 \leq 12 + 12/3 + 3 = 19$ . If  $d|q| = 1$  or  $d|q| = 2$ , then  $S(m, n) \geq S(2, n)$  only for  $m = 1, 2, (n+1)/2$ . For  $|q| \geq 3$ ,  $S(m, n) < S(2, n)$  as soon as  $n/3 + 19 < S(2, n)$ . This is the case for  $n \geq 132$ . Accordingly, Theorem 2 must be checked only for  $n \leq 131$ , where it is also true.  $\square$

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